

THE PARA Fermionic OBSERVABLE IN SLE

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ABSTRACT. The parafermionic observable has recently been used by number of authors to study discrete models, believed to be conformally invariant and to prove convergence results for these processes to SLE [1, 10, 4, 5, 6, 14, 15, 18, 19, 20]. We provide a definition for a one parameter family of continuum versions of the parafermionic observable for SLE, which takes the form of a normalized limit of expressions identical to the discrete definition. We then show the limit defining the observable exists, compute the value of the observable up to a finite multiplicative constant, and prove this constant is non-zero for a wide range of κ . Finally, we show our observable for SLE becomes a holomorphic function for a particular choice of the parameter, which helps illuminate a fundamental property of the discrete observable.

1. INTRODUCTION

The parafermionic observable is a tool which has been used successfully to study discrete models in both the physics [6, 14, 15] and mathematics [1, 3, 4, 5, 18, 19, 20] literature. In particular, the program by Smirnov and coauthors to prove the convergence of various discrete processes to SLE_κ [3, 18, 20] is of great interest. This program has already produced a proof of the convergence of Ising random cluster model boundaries and Ising cluster boundaries to $\text{SLE}_{16/3}$ and SLE_3 , respectively [19]. Recent work by Duminil-Copin and Smirnov in [4] use these same techniques to rigorously compute the connective constant for self-avoiding walk and provide an approach to showing planar self-avoiding walk converges in the scaling limit to $\text{SLE}_{8/3}$.

Central to this program is an object known as the parafermionic observable, which is a one-parameter family, indexed by the *spin* σ , of observables defined by the tuning number of the discrete curves. The fundamental property of this observable is that one may often find a value of σ for which the observable is discretely holomorphic in the end-points of the curve, which allows one to deduce conformal invariance. The reason for the existence of such a σ is not well-understood.

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In this paper, we provide a definition for a similar observable for radial SLE_κ , derived from the discrete definition as reviewed in Section 1.1. We then prove the existence of the observable, compute the value of the observable up to an overall multiplicative constant, and finally show this multiplicative constant is non-zero for a large range of κ by producing an explicit lower bound. In this setting, we may then compute in the continuum the proper value of σ to produce the holomorphic observable, providing some understanding of the value of σ observed in the discrete setting.

We note that a related study was conducted, from a physical perspective, for chordal SLE using a different definition for the continuum parafermionic observable based on the probability that an SLE curve passes to the left of a point [15], and from the point of view of conformal field theory [2].

1.1. The discrete parafermionic observable. Before discussing the continuum generalization of the parafermionic observable, we first review the discrete observable through the example of the self-avoiding walk (SAW) as used in [4]. Let \mathbb{L} denote the hexagonal lattice in \mathbb{C} with unit distance between adjacent points. Fix a bounded domain $\Omega \subseteq \mathbb{C}$ and let $V(\Omega)$ be the set of vertices of $\mathbb{L} \cap \Omega$. We turn $V(\Omega)$ into a graph by adding every half-edge adjacent to any vertex $V(\Omega)$ and mid-edge vertices along every edge. See Figure 1 for an illustration of this setup.

Fix mid-edges z in Ω and w on the boundary of Ω . Given a parameter x , we consider the measure on self avoiding curves starting at w and ending at z on the discretized version on Ω which weighs a curve γ by the weight $x^{\ell(\gamma)}$ where $\ell(\gamma)$ is the number of steps in γ . Thus,

$$F(z) = \mathbb{E}[e^{-i\sigma\mathcal{W}_\gamma(w,z)}] = \sum_{\gamma \subset \Omega: w \rightarrow z} e^{-i\sigma\mathcal{W}_\gamma(w,z)} x^{\ell(\gamma)},$$

where the expectation notation is being used even though the measure need not be a probability measure.

The key result of [4], for the study of SAW, is that for $x = 1/\sqrt{2+\sqrt{2}}$ and $\sigma = \frac{5}{8}$, $F(z)$ satisfies a relation which can be thought of as a (partial) discrete version of the Cauchy-Riemann equation. Moreover, by considering limits of this observable in carefully chosen domains, one may derive that the connective constant is $\sqrt{2+\sqrt{2}}$.

There are two important observations to make about this definition. First, the expectation needs to be taken weighted by the total mass of the curves under the discrete measure and should not be taken under the normalized measure. Second, a key feature of this observable is the existence of a value of σ which makes $F(z)$ holomorphic. These observations help motivate our study of the SLE parafermionic observable.

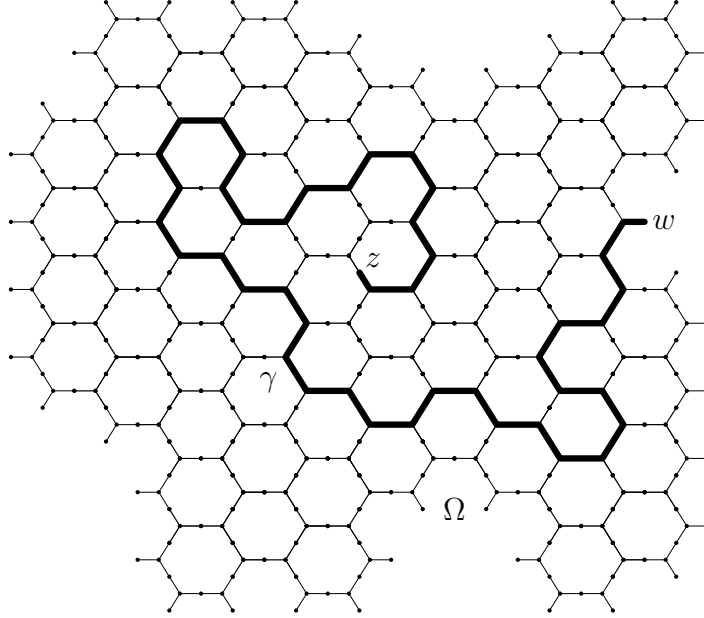


FIGURE 1. An example of a discrete curve γ from w to z in the domain Ω with all vertices and mid-edges illustrated. In this instance the turning number of the curve is $-7\pi/3$.

1.2. Lifting the turning number to the continuum. In the above definition, we required the turning number of the discrete curve—a property only well-defined for piecewise C^1 curves (note that an arbitrary choice must be made at corners; in our case the correct choice is clear). Indeed the turning number of SLE_κ is inherently ill-defined, and thus we must find an alternative quantity which survives in the continuum.

Examine the discrete curve in Figure 1. We consider this curve parametrized continuously to be a curve $\gamma : [0, \infty] \rightarrow \mathbb{C}$ with $\gamma(0) = w$ and $\gamma(\infty) = z$. By the construction of the boundary, the first step of the curve is taken normal to the boundary of the domain. The last step is taken in the direction of $\arg(\gamma'(\infty))$, which is

$$\lim_{t \rightarrow \infty} \arg(z - \gamma(t)).$$

Thus, the turning number is

$$\lim_{t \rightarrow \infty} \arg(z - \gamma(t)) - \arg(\mathbf{n}_\Omega(w)) + 2\pi k$$

for some integer k , where $\mathbf{n}_\Omega(w)$ is the inward facing normal.

To identify the value of k , we exploit the regular homotopy invariance of turning number, which in the discrete setting simply means we may reroute

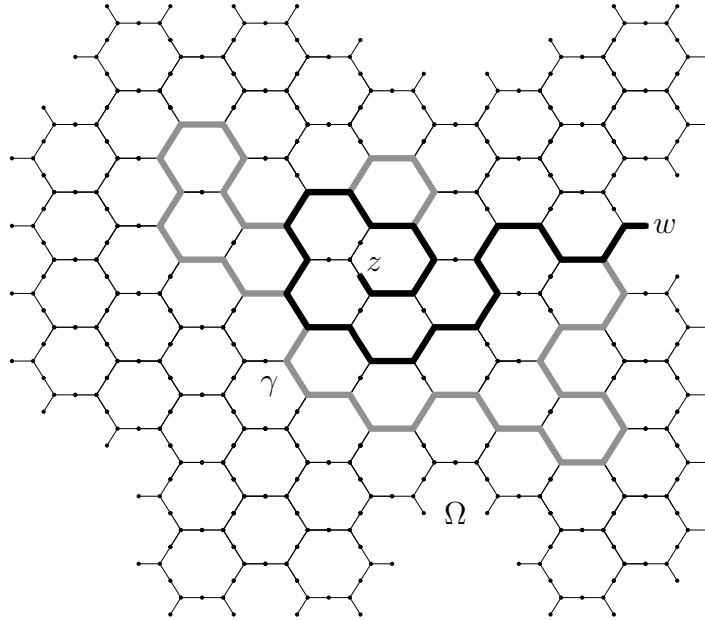


FIGURE 2. The curve γ of Figure 1 is shown here in gray along with a discretely homotopy equivalent curve in black which illustrates that the winding number dictates the correct multiple of 2π .

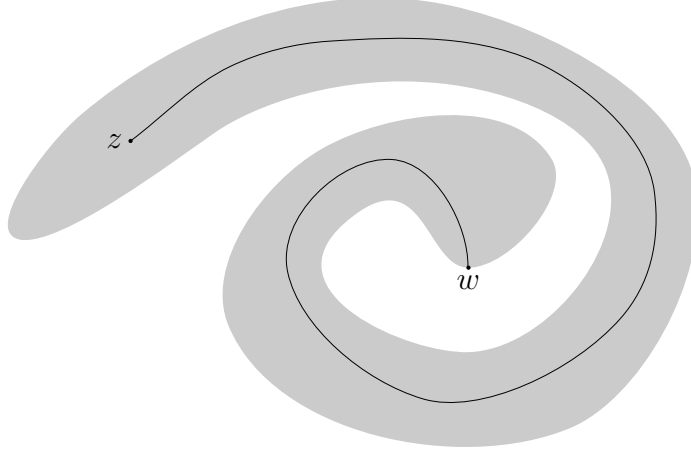
the curve around hexagons as long as doing so does not create any self intersections or move either end. By performing such operations, we may remove any winding which occurs around points other than z without changing the turning number and hence reduce it to a curve as shown in Figure 2. In this case, the correct multiple of 2π is precisely the winding number of γ around z . Taking the version of $\arg(z - \gamma(t))$ which is continuous in t , we thus have $k = 0$ in the above formula.

There is a subtlety in the choice of $\arg(\gamma(0) - z)$ in the above definition. In particular, care must be taken in situations as in Figure 3, where the domain itself adds turning to the curve by winding around w . As we work mainly in \mathbb{D} , the unit disk, this is not an issue.

It will be slightly more convenient to consider $\arg(\gamma(t) - z)$, which differs from the previous choice by π . Thus, if we take $\mathbf{n}_\Omega(w)$ to be the outward facing normal, we find the turning number is

$$\lim_{t \rightarrow \infty} \arg(\gamma(t) - z) - \arg(\mathbf{n}_\Omega(w)).$$

There are two things to note here. First, there is no “first step” of an SLE along the inwards facing normal. However, as this is the case in the discrete

FIGURE 3. A domain with non-trivial winding about w

model, it makes sense to adopt this convention. Second, as an SLE curve approaches the terminal point, the limit in the above formula will not exist—it needs to be properly normalized in the $t \rightarrow \infty$ limit. With these points in mind, we make the following definition.

Definition. Given an arbitrary non-crossing curve $\gamma : [0, \infty] \rightarrow D$ from $w \in \partial D$ to $z \in D$ where ∂D is sufficiently smooth so that the outward facing normal $\mathbf{n}_D(w)$ at w is well-defined, then

$$\mathcal{W}_{\gamma(0,t]}^z = \arg(\gamma(t) - z) - \arg(\mathbf{n}_D(w))$$

where the version of the argument is chosen to be continuous in t and to tend to the correct value as t tends to zero.

An important feature to note about using a normalized limit of $\mathcal{W}_{\gamma(0,t]}^z$ is that this quantity will transform the same as the discrete version under conformal transformations. If one were to apply a conformal transformation to the discrete curve, the turning number is increased by the argument of the derivative at the terminal point and decreased by the argument of the derivative at the initial point. This choice may seem odd from the point of view of conformal field theory since it implies that the observable will be holomorphic at the terminal point and anti-holomorphic at the initial point, however this choice is natural when one attempts to mirror the discrete definition.

1.3. The continuum parafermionic observable. With the notion of turning number generalized, we define the parafermionic observable for SLE_κ by mirroring the discrete construction.

Here, and in the rest of the paper, we will let $a = 2/\kappa$. Consider a radial SLE_κ in \mathbb{D} from 1 to some point z , parametrized so that the conformal radius of z in $D_t := \mathbb{D} \setminus \gamma(-s_0, t]$ is e^{-2at} (the curve starts at some time $-s_0 = -s_0(z)$ since the conformal radius of z in D is not always 1; see Section 2.1 for more detailed discussion). This is (a linear function of) the standard radial parametrization as was used in Schramm's original work defining SLE [17]. Let $\mathcal{W}_{\gamma(-s_0, t]}^z$ be as defined in the previous section. By analogy with the discrete case, we examine the expression

$$\mathbb{E}^z[e^{-i\sigma \mathcal{W}_{\gamma(-s_0, t]}^z}].$$

This expectation differs from the discrete parafermionic observable in two important ways. First, this observable does not take into account the entire curve γ from 1 to z , only the curve up to time t . Second, the observable only considers the SLE probability measure, which is not the correct analogue of the discrete measures encountered in Section 1.1.

To avoid the first issue, we consider the limit as $t \rightarrow \infty$. As t grows, the typical winding number about z grows as well, and hence the expectation shrinks towards zero. In Section 3, we will see the form of normalization needed to avoid triviality of the observable is

$$\lim_{t \rightarrow \infty} e^{2a\nu t} \mathbb{E}^z[e^{-i\sigma \mathcal{W}_{\gamma(-s_0, t]}^z}].$$

for some explicit $\nu = \nu(\kappa, \sigma)$.

To avoid the second issue, we must consider not the probability measure given by SLE_κ , but instead the finite measure given by weighting by the total mass of radial SLE_κ as used by Lawler (see, for example, [9]), which we denote by $C_{\mathbb{D}}(w, z)$ where $w \in \partial\mathbb{D}$ is the initial point of the SLE_κ and $z \in \mathbb{D}$ is the terminal point of the SLE_κ . These weighted measures were defined with the intention of mirroring the properties expected for the total mass of the discrete measures (see [9] for a more detailed exposition). A precise definition is given in Section 2.2.

With this final addition in place, we are ready to state our main object of study.

Definition. The *parafermionic observable for SLE_κ* (in the unit disk from 1 to z) is

$$F(z) = \lim_{t \rightarrow \infty} C_{\mathbb{D}}(1, z) e^{2a\nu t} \mathbb{E}^z[e^{-i\sigma \mathcal{W}_{\gamma(0, t]}^z}],$$

where $\nu = \nu(\kappa, \sigma)$ is chosen so the limit exists.

1.4. Structure of the paper. The paper is arranged as follows.

In Section 2, we provide a brief definition of SLE and the total mass for radial SLE. We then introduce a version of the reverse Loewner equation, our main tool for studying the continuum version of the parafermionic observable,

which is particularly well suited to our task since the turning number is very easily expressed in terms of the flow.

In Section 3, we use this Loewner equation to show the SLE parafermionic observable exists and compute it up to a multiplicative constant (which at this stage of the paper may be zero, but is guaranteed finite). In particular, we show in Theorem 1 that if ν is taken to be $\kappa\sigma^2/2$, then the limit defining $F(z)$ exists, and

$$F(z) = |1 - z|^{-2(b-\sigma)}(1 - |z|^2)^{b-\tilde{b}-\nu}(1 - z)^{-2\sigma}F(0)$$

where $b = \frac{6-\kappa}{2\kappa}$ is the boundary scaling exponent for SLE_κ , and $\tilde{b} = (\frac{\kappa-2}{4})b$ is the interior scaling exponent for SLE_κ . In the process of demonstrating this, we also produce a conformal covariance rule for $F(z)$, which may be found in Theorem 2.

In Section 3.2 we show in Theorem 3 that $F(z)$ is a holomorphic function of z for precisely the choice $\sigma = b$, which agrees with the conjectured limits for the discrete examples discussed in [3, 18, 20].

Finally, in Section 4, we provide conditions on the non-triviality of the observable by exploiting a particularly convenient reparametrization of the reverse Loewner equation. In Theorem 4, we provide a general condition to show non-triviality of $F(z)$ which holds even in the non-holomorphic case. As we are most interested in the holomorphic case, we then specialize it in Theorem 5 to show the holomorphic observable is non-zero for κ approximately in the range $(2.47\dots, 7.99\dots)$, which is defined by a pair of roots to explicit transcendental equations.

2. SLE PRELIMINARIES

We review a few preliminaries about radial SLE_κ . We refer the reader to [8, 9, 22] for more complete introductions to the field.

2.1. Basic definitions. Fix a simply connected domain D , $w \in \partial D$, and $z \in D$. We now describe radial SLE_κ from w to z in D .

Radial SLE_κ is the unique family of probability measures, $\mu_D(w, z)$ on non-crossing curves from w to z contained in D , viewed modulo reparametrization, which satisfy the following two properties:

Conformal Invariance: Given a conformal transformation (by which we mean a conformal map with a conformal inverse) $f : D \rightarrow f(D)$, we have $\mu_{f(D)}(f(w), f(z)) = f \circ \mu_D(w, z)$, where the right hand side is the push-forward of $\mu_D(w, z)$ along f .

Domain Markov: Given $\gamma_t = \gamma(0, t]$, an initial segment of γ , then conditional distribution of $\mu_D(w, z)$ on the remainder of the curve is $\mu_{D \setminus \gamma_t}(\gamma(t), z)$.

These two properties are obtained by examining discrete models, such as the self-avoiding walk. The study of SLE was initiated by Oded Schramm in [17], in which he showed the above properties may be used to derive a much more technically convenient definition, which we now give. Throughout we let $a := 2/\kappa$, and most equations are written in term of a rather than κ .

By conformal invariance, it suffices to only define radial SLE_κ from 1 to 0 in \mathbb{D} . Let g_t be the solution to the initial value problem in \mathbb{D} given by

$$\partial_t g_t(z) = 2ag_t(z) \frac{e^{2iB_t} + g_t(z)}{e^{2iB_t} - g_t(z)}, \quad g_0(z) = z$$

where B_t is a standard Brownian motion, called the *driving function*. This is known as the *radial Loewner equation*. The map g_t defines a conformal transformation from a simply connected domain D_t to \mathbb{D} . Moreover, it is shown in [16] (for $\kappa \neq 8$) and [12] (for $\kappa = 8$) that D_t may be written as the connected component of $\mathbb{D} \setminus \gamma(0, t]$ containing zero with $\gamma(t)$ given by

$$\gamma(t) = \lim_{\delta \downarrow 0} g_t^{-1}((1 - \delta)e^{2iB_t}).$$

This curve is *radial SLE_κ from 1 to 0 in \mathbb{D}* .

Given a simply connected domain D and a point $z \in D$, the *conformal radius* is a measure of distance from z to the boundary defined as $1/|f'(z)|$ where f is a conformal transformation of D to \mathbb{D} sending z to 0. By differentiating in z , we see $\partial_t g'_t(0) = 2ag'_t(0)$ and hence $g'_t(0) = e^{2at}$ which is to say the conformal radius of 0 in D_t is e^{-2at} .

Thus, we have defined radial SLE_κ in \mathbb{D} from 1 to 0 parametrized so the conformal radius of 0 in D_t is e^{-2at} . We may use this to define the curve in other domains by conformal invariance, but due to our application, we need to be careful in our choice of parametrization. We want that for any domain D , our SLE_κ curves from $w \in \partial D$ to $z \in D$ are parametrized so the conformal radius of z in $D \setminus \gamma(-s_0, t]$ is e^{-2at} . In particular if $f : \mathbb{D} \rightarrow D$ with $f(1) = w$ and $f(0) = z$, we reparametrize so that

$$(f \circ \gamma)(t) = f(\gamma(s_t)) \quad \text{where} \quad s_t := t + \frac{1}{2a} \log |f'(0)|.$$

This makes the conformal radius of $f(0)$ in $D \setminus (f \circ \gamma)(-s_0, t]$ equal to e^{-2at} . This has the slight inconvenience that our SLE curves will often be parametrized to start at time 0 but instead at time $-s_0 = -\frac{1}{2a} \log |f'(0)|$.

2.2. Total mass of radial SLE. As stated in Section 1.3, we are not primarily interested in SLE as a probability measure, as defined in the previous section, but instead as a finite measure weighted by what is known as the SLE total mass, which we denote by $C_D(w, z)$.

Throughout, we let $b = (3a - 1)/2$ denote the *boundary scaling exponent*, and $\tilde{b} = \frac{1}{2}(\frac{1}{a} - 1)b$ denote the *interior scaling exponent*. Then, the *total mass*

for radial SLE_κ is defined as follows. For convenience, fix $C_{\mathbb{D}}(1, 0) = 1$. Then, using the covariance rule derived in [9], we get for any conformal map $g : \mathbb{D} \rightarrow g(\mathbb{D})$ with well-defined derivative at $w \in \partial\mathbb{D}$, we have

$$C_{\mathbb{D}}(w, z) = |g'(w)|^b \cdot |g'(z)|^{\bar{b}} \cdot C_{g(\mathbb{D})}(g(w), g(z)).$$

In this work, we are particularly interested in the case $C_{\mathbb{D}}(1, z)$. Consider the conformal map $f_z : \mathbb{D} \rightarrow \mathbb{D}$ which is the unique conformal automorphism of \mathbb{D} for which $f_z(z) = 0$ and $f_z(1) = 1$. As this function is useful several times in our paper, we collect a few properties of it in a lemma.

Lemma 2.1. *For $f_z : \mathbb{D} \rightarrow \mathbb{D}$ as defined above, we have the following statements:*

$$\begin{aligned} f_z(w) &= \frac{1 - \bar{z}}{1 - z} \cdot \frac{w - z}{1 - \bar{z}w}, \\ f'_z(w) &= \frac{1 - \bar{z}}{1 - z} \cdot \frac{1 - |z|^2}{(1 - \bar{z}w)^2}, \\ |f'_z(1)| &= \frac{1 - |z|^2}{|1 - z|^2}, \\ |f'_z(z)| &= \frac{1}{1 - |z|^2}, \\ \arg(f'_z(1)) &= 0, \text{ and} \\ \arg(f'_z(z)) &= \arg\left(\frac{1 - \bar{z}}{1 - z}\right) = -2 \arg(1 - z). \end{aligned}$$

Proof. The particular form of f_z follows since every conformal automorphism of the disk, f , is of the form

$$f(w) = e^{i\theta} \frac{w - z}{1 - \bar{z}w}$$

for some $z \in \mathbb{D}$ and $\theta \in \mathbb{R}$ (see, for example, [21]). With the explicit form available, the rest is computation. ■

By applying the conformal covariance rule for the total mass, we see

$$C_{\mathbb{D}}(1, z) = |f'_z(1)|^b \cdot |f'_z(z)|^{\bar{b}} \cdot C_{\mathbb{D}}(1, 0) = |1 - z|^{-2b} (1 - |z|^2)^{b - \bar{b}}.$$

2.3. Alternate forms of the Loewner equation and the turning number. To prove our theorem, we work with a lifted form of the radial Loewner equation and a corresponding reversed equation. Let g_t be the solution to the initial value problem

$$\partial_t g_t(z) = 2a g_t(z) \frac{e^{2iB_t} + g_t(z)}{e^{2iB_t} - g_t(z)}, \quad g_0(z) = z$$

as in the previous section.

We lift this equation to the upper half plane via the same method as [11]. Define a function $h_t(z)$ by requiring

$$g_t(e^{2iz}) = e^{2ih_t(z)}.$$

This does not define $h_t(z)$ uniquely, but by translating the Loewner equation to an equation on $h_t(z)$, we have

$$\partial_t h_t(z) = a \cot(h_t(z) - B_t)$$

as long as we choose $h_t(z)$ to be continuous in t . We can make $h_t(z)$ uniquely defined by choosing $h_0(z) = z$ as the initial conditions.

Consider h_t^{-1} . By the definition of the curve generated by the Loewner chain, we see that

$$\gamma(t) = \lim_{\delta \downarrow 0} g_t^{-1}((1 - \delta)e^{2iB_t}) = \lim_{\delta \downarrow 0} e^{2ih_t^{-1}(B_t + i\delta)}$$

where the limit exists for all t with probability one. Thus if one is interested in the winding number, we need to consider

$$\mathcal{W}_{\gamma(0,t]}^0 = \arg(\gamma(t)) - \arg(\mathbf{n}_D(1)) = \lim_{\delta \downarrow 0} 2\operatorname{Re}(h_t^{-1}(B_t + i\delta))$$

where each step must interpreted as the version which is continuous in t . As the right hand side is already continuous in t with the correct value of 0 at $t = 0$, we need not worry further.

When one wishes to study the behavior of the inverse of a Loewner map, it has proven to be useful to consider the reverse Loewner flow (see, for example, [16, 7]). In our setting, this is a process Z_t satisfying

$$dZ_t(z) = -a \cot(Z_t(z)) dt + dB_t, \quad Z_0(z) = z.$$

This is related to h_t^{-1} as follows.

Lemma 2.2. *For a fixed $t > 0$, the random conformal maps $h_t^{-1}(B_t + \cdot)$ and $Z_t(\cdot)$ are equal in law.*

The proof of this lemma is identical to the equivalent statement for the chordal case, which may be found, for example, in [7, Lemma 5.2], and is thus omitted. It is important to note this only holds for any *fixed* $t > 0$ and not on the process level.

For simplicity of notation, let us write $Z_t(z) = \Theta_t(z) + iR_t(z)$. Thus we have that, in law,

$$\mathcal{W}_{\gamma(0,t]}^0 = \lim_{\delta \downarrow 0} 2\Theta_t(i\delta).$$

In this setup, to analyze the turning number we need only analyze the real part of the reverse flow.

By the definition of $\cot(z)$ we have

$$\cot(x + iy) = i \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{e^{i(x+iy)} - e^{-i(x+iy)}} = \frac{\sin(2x)}{\cosh(2y) - \cos(2x)} - i \frac{\sinh(2y)}{\cosh(2y) - \cos(2x)}$$

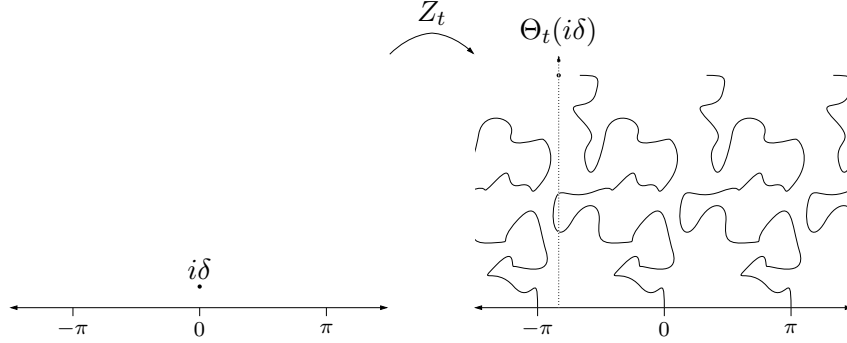


FIGURE 4. A diagram illustrating the lifted reverse radial Loewner equation Z_t . This is a conformal map from \mathbb{H} to a subdomain of \mathbb{H} slit by repeated translates of curves growing out of 0 (and its translates). While convenient for comparing with the turning number, this makes the growth of Θ_t erratic.

and thus by applying Itô's formula

$$d\Theta_t = -a \frac{\sin(2\Theta_t)}{\cosh(2R_t) - \cos(2\Theta_t)} dt + dB_t,$$

$$dR_t = a \frac{\sinh(2R_t)}{\cosh(2R_t) - \cos(2\Theta_t)} dt.$$

We now split Θ_t into two parts by defining T_t so that $\Theta_t = T_t + B_t$. This choice is motivated by the following geometric interpretation of the reverse flow, which is fundamental to the intuition behind the work that follows.

Consider the growth of Θ_t . This should be thought of as the total motion of the lifted SLE_κ curve in the real direction, which, as seen above, corresponds to the turning number. Its value should become large, and thus its variance hard to control. Figure 4 illustrates the motion of Θ_t .

However, now consider $T_t = \Theta_t - B_t$. This translation has the effect of growing the curves from translates of B_t rather than from translates of 0. In this case, the majority of the change in argument has been shifted to change in location of the base of growth of the curve, whereas the tip, which is traced by T_t , grows mostly vertically. Indeed, by our definition, we have that

$$dT_t = -a \frac{\sin(2\Theta_t)}{\cosh(2R_t) - \cos(2\Theta_t)} dt.$$

which implies the rate of displacement of T_t decays exponentially in R_t . The exponential decay hints at what we prove rigorously in the next section: T_t has a limit as $t \rightarrow \infty$. Figure 5 illustrates the motion of T_t .

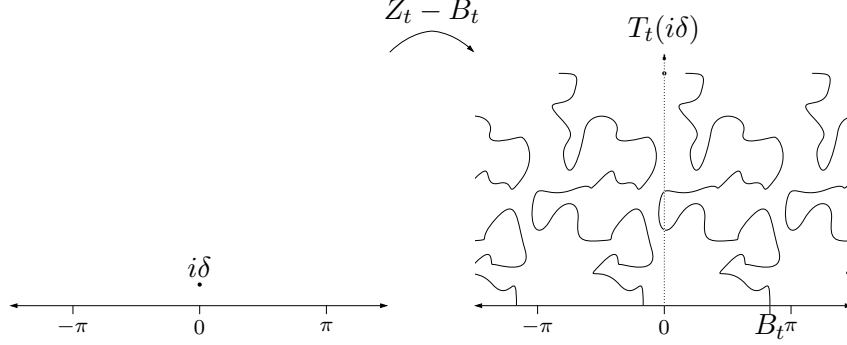


FIGURE 5. This illustrates the growth of the process T_t . As opposed to Figure 4, the new segments of the curve are being added at shifting locations along \mathbb{R} . This has the effect of removing motion of the tip caused by B_t , making the tip travel essentially vertically (the effect is exaggerated in this diagram).

Thus, by splitting Θ_t into $T_t + B_t$ we have translated understanding the turning number into understanding the growth of a Brownian term and the growth of a term which has a $t \rightarrow \infty$ limit.

3. THE EXISTENCE OF THE SLE PARA-FERMIONIC OBSERVABLE

3.1. **The existence of the limit.** We wish to show

$$\frac{F(z)}{C_{\mathbb{D}}(1, z)} = \lim_{t \rightarrow \infty} e^{2avt} \mathbb{E}^z[e^{-i\sigma\mathcal{W}_{\gamma(0,t)}^z}]$$

exists when $\nu = \sigma^2/a$. We do so by first showing it exists when $z = 0$, and then deriving a conformal covariance rule to see the limit exists for all z , and moreover computing it as a function of z .

Fixing $z = 0$ (and, for simplicity, suppressing it in our notation), we may use the reverse flow facts from above. Thus we need to show

$$\lim_{t \rightarrow \infty} e^{2avt} \mathbb{E}[e^{-i\sigma\mathcal{W}_{\gamma(0,t)}}] = \lim_{t \rightarrow \infty} \lim_{\delta \downarrow 0} e^{2avt} \mathbb{E}[e^{-2i\sigma T_t(i\delta)} e^{-2i\sigma B_t}]$$

exists (where the location of the $\delta \rightarrow 0$ limit is justified by the dominated convergence theorem). As a first step we have the following.

Proposition 3.1. *We have*

$$\lim_{t \rightarrow \infty} e^{2avt} \mathbb{E}[e^{-i\sigma\mathcal{W}_{\gamma(0,t)}}] = \lim_{t \rightarrow \infty} \lim_{\delta \downarrow 0} \mathbb{E}[e^{-2i\sigma T_t(i\delta)}].$$

Proof. The case $\sigma = 0$ is trivial, and so we need only provide a proof in the case $\sigma \neq 0$. By the Cauchy-Schwartz inequality, we have

$$\begin{aligned} |\mathbb{E}[e^{-2i\sigma T_t(i\delta)} e^{-2i\sigma B_t}] - \mathbb{E}[e^{-2i\sigma T_t(i\delta)}] \mathbb{E}[e^{-2i\sigma B_t}]| &= |\text{Cov}(e^{-2i\sigma T_t(i\delta)}, e^{-2i\sigma B_t})| \\ &\leq (\mathbb{E}[e^{-4i\sigma B_t}])^{1/2} \\ &= e^{-4\sigma^2 t} \end{aligned}$$

where the second line follows since the variance of $e^{-2i\sigma T_t(i\delta)}$ is no greater than one, and the third line follows by the calculation of the characteristic function of a normal random variable. Written another way, this states

$$\begin{aligned} \mathbb{E}[e^{-2i\sigma T_t(i\delta)} e^{-2i\sigma B_t}] &= \mathbb{E}[e^{-2i\sigma T_t(i\delta)}] \mathbb{E}[e^{-2i\sigma B_t}] + O(e^{-4\sigma^2 t}) \\ &= e^{-2\sigma^2 t} \mathbb{E}[e^{-2i\sigma T_t(i\delta)}] + O(e^{-4\sigma^2 t}) \end{aligned}$$

as $t \rightarrow \infty$ where the implicit constant is one, and hence does not depend on δ .

Thus, we have

$$\begin{aligned} e^{2avt} \mathbb{E}[e^{-2i\sigma T_t(i\delta)} e^{-2i\sigma B_t}] &= e^{2avt-2\sigma^2 t} \mathbb{E}[e^{-2i\sigma T_t(i\delta)}] + O(e^{2avt-4\sigma^2 t}) \\ &= \mathbb{E}[e^{-2i\sigma T_t(i\delta)}] + O(e^{-2\sigma^2 t}) \end{aligned}$$

by the definition of ν , where the implicit constant is again one, and hence independent of δ . As $\sigma^2 > 0$, this gives the desired result. \blacksquare

Thus, we have reduced the existence of the limit to controlling $T_t(i\delta)$. We will show

$$T_\infty := \lim_{t \rightarrow \infty} \lim_{\delta \downarrow 0} T_t(i\delta)$$

exists with probability one. This suffices, by the dominated convergence theorem, to show the limit defining $F(0)$ exists.

Consider the process

$$T_t(0) := \lim_{\delta \downarrow 0} T_t(i\delta).$$

As mentioned in Section 2.1, for a fixed time $t \geq 0$, this limit exists due to the existence of the SLE_κ curve. To study $T_t(0)$ (and related reverse flow processes), we use the following lemma, stating that the limit versions as above all satisfy their equivalent SDEs and ODEs.

Lemma 3.2. *The processes $R_t(0)$, $T_t(0)$, $\Theta_t(0)$, and $Z_t(0)$ all satisfy the ODEs and SDEs satisfied by $R_t(i\delta)$, $T_t(i\delta)$, $\Theta_t(i\delta)$, and $Z_t(i\delta)$ respectively for all $t > 0$ with probability one.*

Proof. Fix an arbitrary $t > 0$. First note $R_t(0) > 0$ with probability one since in distribution $R_t(0) = \log |\gamma(t)|$ and $\gamma(t) \in \mathbb{D}$ with probability one. This is immediate for $\kappa \leq 4$ as such SLE_κ curves never hit the boundary with probability one [16, Theorem 6.1]. For $\kappa > 4$, one may see this by observing that the equivalent statement holds for chordal SLE by scaling and then using

absolute continuity of radial SLE with chordal SLE (see, for example, [9]). This implies

$$d(T_t(0) + iR_t(0)) = d(Z_t(0) - B_t) = -a \cot(Z_t(0)) dt$$

is bounded and continuous in a neighborhood of $T_t(0) + iR_t(0)$ and thus we have

$$\frac{dT_t(0)}{dt} = \lim_{\delta \downarrow 0} \frac{dT_t(i\delta)}{dt} \quad \text{and} \quad \frac{dR_t(0)}{dt} = \lim_{\delta \downarrow 0} \frac{dR_t(i\delta)}{dt}.$$

This holds with probability one for any fixed t , and thus it holds for all rational t . We may conclude that it holds for all $t > 0$ by continuity of the derivative of $T_t(i\delta)$ and $R_t(i\delta)$.

To extend this claim to $Z_t(0)$ and $\Theta_t(0)$, simply note these are sums of $T_t(0)$, $R_t(0)$ and B_t . ■

We may now prove T_∞ exists.

Proposition 3.3. $T_\infty := \lim_{t \rightarrow \infty} T_t(0)$ exists with probability one.

Proof. Let $\tau = \inf\{t > 0 \mid R_t(0) \geq \log(\sqrt{3})\}$. We first show this stopping time is finite with probability one. By Lemma 2.2, we know

$$Z_t(\cdot) = h_t^{-1}(B_t + \cdot)$$

in distribution. Since

$$\gamma(t) = \lim_{\delta \downarrow 0} e^{2ih_t^{-1}(B_t + i\delta)},$$

we obtain

$$\gamma(t) = e^{2i\Theta_t(0) - 2R_t(0)}$$

in distribution. Since $R_t(0)$ is non-decreasing in t (by the ODE it satisfies),

$$\begin{aligned} \mathbb{P}\{\tau < \infty\} &= \lim_{t \rightarrow \infty} \mathbb{P}\{\tau < t\} \\ &= \lim_{t \rightarrow \infty} \mathbb{P}\{R_t(0) \geq \log(\sqrt{3})\} \\ &= \lim_{t \rightarrow \infty} \mathbb{P}\left\{|\gamma(t)| \leq \frac{1}{3}\right\} \\ &= 1, \end{aligned}$$

where the last line holds since with probability one $\lim_{t \rightarrow \infty} \gamma(t) = 0$ (see [11]).

Since $\tau < \infty$ with probability one, $T_\tau(0)$ is well-defined. We now use the ODEs satisfied by $R_t(0)$ and $T_t(0)$ to show that for all $t > \tau$,

$$\left| \frac{dT_t(0)}{dt} \right| \leq \frac{2a}{3} e^{-at},$$

and hence we have $\lim_{t \rightarrow \infty} T_t(0)$ exists with probability one, and indeed cannot differ by more than a constant from $T_\tau(0)$.

First, we bound below the rate of growth of $R_t(0)$ after τ . For any $t > \tau$, we have

$$\begin{aligned} \frac{dR_t(0)}{dt} &= a \frac{\sinh(2R_t(0))}{\cosh(2R_t(0)) - \cos(2\Theta_t(0))} \\ &= a \frac{1 - e^{-4R_t(0)}}{1 + e^{-4R_t(0)} - 2e^{-2R_t(0)} \cos(2\Theta_t(0))} \\ &\geq a \frac{1 - e^{-4R_t(0)}}{1 + e^{-4R_t(0)} + 2e^{-2R_t(0)}} \\ &\geq \frac{a}{2} \end{aligned}$$

where the last line holds by our definition of τ . Thus we have

$$R_{\tau+t}(0) \geq \log(\sqrt{3}) + \frac{a}{2}t.$$

We now turn our attention to $T_t(0)$. For any $t > \tau$, we have

$$\begin{aligned} \left| \frac{dT_{\tau+t}(0)}{dt} \right| &= \left| a \frac{\sin(2\Theta_{\tau+t}(0))}{\cosh(2R_{\tau+t}(0)) - \cos(2\Theta_{\tau+t}(0))} \right| \\ &\leq \frac{2a}{e^{2R_{\tau+t}(0)} + e^{-2R_{\tau+t}(0)} - \cos(2\Theta_{\tau+t}(0))} \\ &\leq 2ae^{-2R_{\tau+t}(0)} \\ &\leq 2ae^{-2(\log(\sqrt{3}) + \frac{a}{2}t)} \\ &= \frac{2a}{3}e^{-at} \end{aligned}$$

where we have used our bound on $R_{\tau+t}(0)$ in the second to last line. Thus we have proven the limit exists with probability one. \blacksquare

Thus, by the dominated convergence theorem, we have completed our proof of the following.

Proposition 3.4. *The limit*

$$F(0) = C_{\mathbb{D}}(1, 0) \lim_{t \rightarrow \infty} e^{2a\nu t} \mathbb{E}^0[e^{-i\sigma \mathcal{W}_{\gamma(0,t]}^0}]$$

exists when $\nu = \sigma^2/a$, and moreover

$$F(0) = \mathbb{E}^0[e^{-2i\sigma T_\infty}],$$

which implies $0 \leq |F(0)| \leq 1$.

We now extend this to show the limit defining $F(z)$ exists for all $z \in \mathbb{D}$. Consider f_z as defined in Lemma 2.1. We have

$$\begin{aligned} F(z) &= C_{\mathbb{D}}(1, z) \lim_{t \rightarrow \infty} e^{2avt} \mathbb{E}^z[e^{-i\sigma \mathcal{W}_{\gamma(-s_0, t]}^z}] \\ &= C_{\mathbb{D}}(1, z) \lim_{t \rightarrow \infty} e^{2avt} \mathbb{E}^0[e^{-i\sigma \mathcal{W}_{(f_z^{-1} \circ \gamma)(-s_0, t]}^z}] \end{aligned}$$

by conformal invariance and our choice of parametrization of $(f_z^{-1} \circ \gamma)$. Thus, we need to understand $\mathcal{W}_{(f_z^{-1} \circ \gamma)(-s_0, t]}^z$.

Consider any curve $\gamma : [0, \infty] \rightarrow \mathbb{D}$ with $\gamma(0) = 1$ and $\gamma(\infty) = 0$ and, to avoid pathologies, $0 \notin \gamma(0, \infty)$ (as is the case for radial SLE curves from 1 to 0 in \mathbb{D}). For any conformal transformation f with well-defined derivative at 1, we consider $f \circ \gamma$ as being reparametrized by

$$(f \circ \gamma)(t) = f(\gamma(s_t)) \quad \text{where} \quad s_t := t + \frac{1}{2a} \log |f'(0)|$$

as we have done with our SLE curves. By Taylor expanding the logarithm around z , we get

$$\begin{aligned} \mathcal{W}_{(f \circ \gamma)(-s_0, t]}^z &= \arg((f \circ \gamma)(t) - f(0)) - \arg(\mathbf{n}_f(\mathbb{D})(f(1))) \\ &= \arg(f(\gamma(s_t)) - f(0)) - \arg(f'(1)) \\ &= \arg(f'(0)) + \arg(\gamma(s_t)) - \arg(f'(1)) + O(|\gamma(s_t)|) \\ &= \mathcal{W}_{\gamma(0, s_t]}^0 + \arg(f'(0)) - \arg(f'(1)) + O(|\gamma(s_t)|) \end{aligned}$$

for $|\gamma(s_t)|$ sufficiently small, where the implicit constant depends only on f . Again, care must be taken so the arguments are chosen to be continuous, which in this case is to say $\arg(f'(z))$ is continuous in \mathbb{D} . This establishes the following.

Lemma 3.5. *Let $f : \mathbb{D} \rightarrow f(\mathbb{D})$ be a conformal map with $f'(1)$ well-defined. Then*

$$\mathcal{W}_{(f \circ \gamma)(-s_0, t]}^{f(0)} = \mathcal{W}_{\gamma(0, s_t]}^0 + \arg(f'(0)) - \arg(f'(1)) + O(|\gamma(s_t)|).$$

as $t \rightarrow \infty$ where the implicit constant depends only on f .

We now return to our specific case. By using Lemma 3.5 and Lemma 2.1, we see that

$$\begin{aligned}
F(z) &= C_{\mathbb{D}}(1, z) \lim_{t \rightarrow \infty} e^{2a\nu t} \mathbb{E}^0[e^{-i\sigma \mathcal{W}_{(f_z^{-1} \circ \gamma)(-s_0, t]}^z}] \\
&= C_{\mathbb{D}}(1, z) \lim_{t \rightarrow \infty} e^{2a\nu t} \mathbb{E}^0[e^{-i\sigma[\mathcal{W}_{\gamma(0, s_t]}^0 - \arg(f'_z(z)) + \arg(f'_z(1)) + O(|\gamma(s_t)|)l}]] \\
&= C_{\mathbb{D}}(1, z) e^{\nu \log |f'_z(z)|} e^{i\sigma \arg(f'_z(z))} \lim_{t \rightarrow \infty} e^{2a\nu s_t} \mathbb{E}^0[e^{-i\sigma[\mathcal{W}_{\gamma(0, s_t]}^0 + O(|\gamma(s_t)|)l}]] \\
&= C_{\mathbb{D}}(1, z) |f'_z(z)|^\nu e^{-2i\sigma \arg(1-z)} \lim_{t \rightarrow \infty} e^{2a\nu s_t} \mathbb{E}^0[e^{-i\sigma[\mathcal{W}_{\gamma(0, s_t]}^0 + O(|\gamma(s_t)|)l}]] \\
&= C_{\mathbb{D}}(1, z) (1 - |z|^2)^{-\nu} e^{-2i\sigma \arg(1-z)} F(0) \\
&= |1 - z|^{-2b} (1 - |z|^2)^{b-\tilde{b}-\nu} e^{-2i\sigma \arg(1-z)} F(0) \\
&= |1 - z|^{-2(b-\sigma)} (1 - |z|^2)^{b-\tilde{b}-\nu} (1 - z)^{-2\sigma} F(0).
\end{aligned}$$

The third to last line is non-trivial to justify; however, the proof follows the same argument used to prove the existence of the limit. That is: first rewrite $\mathcal{W}_{\gamma(0, s_t]}^0$ in terms of B_t and T_t , second use Cauchy-Schwartz to cancel the Brownian term with the $e^{2a\nu s_t}$ term, and third use dominated convergence to move the limits inside the expectation.

This yields the following theorem.

Theorem 1. *When $\nu = \sigma^2/a$, the limit defining $F(z)$ exists for all z and moreover*

$$F(z) = |1 - z|^{-2(b-\sigma)} (1 - |z|^2)^{b-\tilde{b}-\nu} (1 - z)^{-2\sigma} F(0).$$

The same argument above can, with only a change of notation, yield the following conformal covariance rule.

Theorem 2. *Given a domain D which is C^1 near a boundary point $w \in \partial D$ and a point z in the interior. Define*

$$F_D(w, z) := C_D(w, z) \lim_{t \rightarrow \infty} e^{2a\nu t} \mathbb{E}^z[e^{-i\sigma \mathcal{W}_{\gamma(-s_0, t]}^z}].$$

Let $f : D \rightarrow f(D)$ be a conformal transformation with $f'(w)$ well-defined, then

$$F_{f(D)}(f(w), f(z)) = |f'(z)|^{\sigma-\tilde{b}-\nu} |f'(w)|^{-b-\sigma} f'(z)^{-\sigma} f'(w)^\sigma F_D(w, z).$$

3.2. Observations on Theorem 1, and the conformal observable. We wish to make a few observations on the above results.

First, the result above does not ensure that $F(z)$ is not uniformly zero in all of \mathbb{D} . To prove it is not, at least for a range of κ , we engage in a careful analysis of the reverse flow, which is the topic of the majority of the remainder of the paper.

Second, recall from the discrete setup that we are most interested in the case where $F(z)$ is a holomorphic function of z .

Theorem 3. *For any a , there is a unique choice of σ which makes $F(z)$ a holomorphic function of z . In particular, for $\sigma = b$, we have*

$$F(z) = (1 - z)^{-2b} F(0)$$

and moreover it satisfies the conformal covariance rule (under the same conditions as Theorem 2)

$$F_{f(D)}(f(w), f(z)) = f'(z)^{-b} \overline{f'(w)}^{-b} F_D(w, z).$$

Proof. Consider the form of $F(z)$ from Theorem 1. It is a product of $F(0)$, some arbitrary complex constant, $(1 - z)^{-2\sigma}$, a holomorphic function, and

$$|1 - z|^{-2(b-\sigma)} (1 - |z|^2)^{b-\tilde{b}-\nu},$$

a real valued function. Thus, for $F(z)$ to be holomorphic, the real-valued function must be constant. By examining the function in a neighborhood of 1, we obtain the necessary condition $\sigma = b$. Direct computation shows the function to be constant for this choice. \blacksquare

This result agrees with what one would expect from the discrete theory (see [3, 18, 20]). Indeed if we return to the self-avoiding walk from Section 1.1, we see the value of σ computed here agrees with what one would expect if the self-avoiding walk were to converge to $\text{SLE}_{8/3}$ (both yielding $\sigma = \frac{5}{8}$).

It is also worth stating that all results proven thus far work for all values of κ , even in the $\kappa \geq 8$ case where the curves are spacefilling (see, [16, Section 6] or [9, Theorem 3.36] for a discussion of the geometric phases of SLE_κ). In the following sections, where we provide conditions to ensure $F(0) \neq 0$, this is no longer the case, and we are only able to provide non-trivial bounds for $\kappa < 8$.

A final comment of note on this computation is to draw attention to very surprising coincidence required for this proof to work: that $b - \tilde{b} - \nu = 0$ when $\sigma = b$. In this computation, there is no *a priori* reason that this should occur—it simply does. Equally intriguing is that the same computation occurs in the recent work of Cardy despite a distinct point of view [2]. This suggests that there might be a deeper interpretation of this relationship than the mere cancelation observed here. Stated another way: the choice $\sigma = b$ can be seen directly from the definition to be required to ensure anti-holomorphicity in the boundary point, whereas the understanding of why this implies holomorphicity in the interior point remains formal.

4. NON-TRIVIALITY OF THE SLE PARA-FERMIONIC OBSERVABLE

We wish to show

$$F(0) = \mathbb{E}[e^{-2i\sigma T_\infty}] = \lim_{t \rightarrow \infty} \lim_{\delta \rightarrow 0} \mathbb{E}[e^{-2i\sigma T_t(i\delta)}]$$

does not vanish for a large range of κ . To do so, we need to analyze the lifted form of the reverse radial Loewner equation from Section 2.3. We make use of the following method.

Lemma 4.1. *For a symmetric random variable X , we have*

$$\mathbb{E}[e^{-i\sigma X}] \geq 1 - \frac{1}{2}\sigma^2 \mathbb{E}[X^2].$$

Proof. Since X is symmetric we know the expectation is real and hence

$$\mathbb{E}[e^{-i\sigma X}] = \mathbb{E}[\cos(\sigma X)] \geq 1 - \frac{1}{2}\sigma^2 \mathbb{E}[X^2].$$

■

Thus, to bound $F(0)$ away from zero we bound $\mathbb{E}[T_t(i\delta)^2]$ above uniformly in t and δ . It is much simpler to work with a reparametrized version of this process in which R_t increases linearly. In the next three subsections, we perform this reparametrization, bound the reparametrized process, and finally transfer this bound back to the originally parametrized process.

For notational simplicity, we let $\|X\|_2 := \mathbb{E}[X^2]^{1/2}$.

4.1. The key time change. Recall the equation

$$dZ_t(z) = -a \cot(Z_t(z)) dt + dB_t.$$

We deal exclusively with $Z_t(i\delta)$ and thus suppress the explicit δ dependence.

Recall $Z_t = \Theta_t + iR_t$ where Θ_t and R_t satisfy

$$\begin{aligned} d\Theta_t &= -a \frac{\sin(2\Theta_t)}{\cosh(2R_t) - \cos(2\Theta_t)} dt + dB_t, \\ dR_t &= a \frac{\sinh(2R_t)}{\cosh(2R_t) - \cos(2\Theta_t)} dt. \end{aligned}$$

As noted in our proof of the existence of the limit in Proposition 3.3, R_t is increasing and for sufficiently large R_t the rate of increase is $a + O(e^{-R_t})$. To take advantage of this regularity, we reparametrize this process by a time change $\tau(t)$ so that $R_{\tau(t)} = t + \delta$.

For a fixed $\delta > 0$, this time change τ is very well-behaved. Indeed, it only differs from a linear time change by at most a δ dependent additive constant.

Lemma 4.2. *For any δ we have*

$$\sinh^{-1}(\sinh(\delta)e^{at}) \leq R_t(i\delta) \leq \cosh^{-1}(\cosh(\delta)e^{at})$$

and hence

$$at + \log(2 \sinh(\delta)) - \delta \leq \tau^{-1}(t) \leq at + \log(2 \cosh(\delta)) - \delta$$

and

$$\frac{1}{a}t + \frac{1}{a}(\delta - \log(2 \sinh(\delta))) \leq \tau(t) \leq \frac{1}{a}t + \frac{1}{a}(\delta - \log(2 \cosh(\delta))).$$

Proof. We have

$$\frac{dR_t}{dt} = a \frac{\sinh(2R_t)}{\cosh(2R_t) - \cos(2\Theta_t)}$$

and hence R_t is increasing and

$$a \frac{\sinh(2R_t)}{\cosh(2R_t) + 1} \leq \frac{dR_t}{dt} \leq a \frac{\sinh(2R_t)}{\cosh(2R_t) - 1}.$$

The solutions to these ODEs with $R_0 = \delta$ are

$$\sinh^{-1}(\sinh(\delta)e^{at}) \quad \text{and} \quad \cosh^{-1}(\cosh(\delta)e^{at})$$

giving the desired bounds on R_t .

Now, recall $R_{\tau(t)} = t + \delta$ by definition of $\tau(t)$, or equivalently $R_t = \tau^{-1}(t) + \delta$. Combining this with the bounds

$$\cosh^{-1}(x) \leq \log(2x) \quad \text{and} \quad \sinh^{-1}(x) \geq \log(2x)$$

gives the desired bounds on $\tau^{-1}(t)$, which may be rearranged to give the bounds on $\tau(t)$. ■

Given any process A_t , we adopt the notation $\hat{A}_t = A_{\tau(t)}$.

Lemma 4.3. *We have*

$$\frac{d\tau}{dt} = \frac{\cosh(2(t + \delta)) - \cos(2\hat{\Theta}_t)}{a \sinh(2(t + \delta))},$$

and thus

$$d\hat{\Theta}_t = -\frac{\sin(2\hat{\Theta}_t)}{\sinh(2(t + \delta))} dt + \left(\frac{\cosh(2(t + \delta)) - \cos(2\hat{\Theta}_t)}{a \sinh(2(t + \delta))} \right)^{1/2} dW_t$$

and

$$d\hat{T}_t = -\frac{\sin(2\hat{\Theta}_t)}{\sinh(2(t + \delta))} dt.$$

Proof. By definition of $\tau(t)$,

$$\int_0^{\tau(t)} a \frac{\sinh(2R_s)}{\cosh(2R_s) - \cos(2\Theta_s)} ds = t,$$

and thus

$$\frac{d\tau}{dt} = \frac{\cosh(2(t + \delta)) - \cos(2\hat{\Theta}_t)}{a \sinh(2(t + \delta))}.$$

We may apply this time change to Θ_t to see

$$\begin{aligned} d\hat{\Theta}_t &= -\frac{\sin(2\hat{\Theta}_t)}{\sinh(2(t+\delta))} dt + d\hat{B}_t \\ &= -\frac{\sin(2\hat{\Theta}_t)}{\sinh(2(t+\delta))} dt + \left(\frac{\cosh(2(t+\delta)) - \cos(2\hat{\Theta}_t)}{a \sinh(2(t+\delta))} \right)^{1/2} dW_t. \end{aligned}$$

The claimed statement for T_t now follows. ■

4.2. The bound on $\mathbb{E}[\hat{T}_t^2]$ and $\mathbb{E}[T_t^2]$. Our argument proceeds as follows. From Lemma 4.3, we know

$$d\hat{T}_t = -\frac{\sin(2\hat{\Theta}_t)}{\sinh(2(t+\delta))} dt.$$

This ODE is well-behaved for t large due to the exponential decay provided by $\sinh(t+\delta)$. Thus, we must concentrate on controlling the ODE for small times. In this case, the only term which can provide help is $\sin(2\hat{\Theta}_t)$. Thus, we must bound the growth of $\hat{\Theta}_t$. We do so by deriving a differential inequality which bounds the variance of $\hat{\Theta}_t$. We then may produce a bound on the growth of the variance of \hat{T}_t for small times t which when combined with the exponential decay gives us a uniform bound the variance of \hat{T}_t for all t and δ .

By Lemma 4.3, we have

$$d\hat{\Theta}_t = -\frac{\sin(2\hat{\Theta}_t)}{\sinh(2(t+\delta))} dt + \left(\frac{\cosh(2(t+\delta)) - \cos(2\hat{\Theta}_t)}{a \sinh(2(t+\delta))} \right)^{1/2} dW_t$$

and hence

$$\begin{aligned} d\hat{\Theta}_t^2 &= \left[\frac{\cosh(2(t+\delta)) - \cos(2\hat{\Theta}_t)}{a \sinh(2(t+\delta))} - \frac{2\hat{\Theta}_t \sin(2\hat{\Theta}_t)}{\sinh(2(t+\delta))} \right] dt \\ &\quad + 2\hat{\Theta}_t \left(\frac{\cosh(2(t+\delta)) - \cos(2\hat{\Theta}_t)}{a \sinh(2(t+\delta))} \right)^{1/2} dW_t. \end{aligned}$$

Since the drift and diffusion coefficients in the SDE for $\hat{\Theta}_t$ are uniformly bounded for any fixed $\delta > 0$, we have for any $S > 0$

$$\int_0^S \mathbb{E}[\hat{\Theta}_s^2] ds < \infty$$

and thus the same property holds for the diffusion coefficient of the SDE for $\hat{\Theta}_t^2$. This is sufficient to imply the diffusion component is a martingale (see,

for example, [13, Corollary 3.2.6]), and thus

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[\hat{\Theta}_t^2] &= \frac{\cosh(2(t+\delta)) - \mathbb{E}[\cos(2\hat{\Theta}_t)]}{a \sinh(2(t+\delta))} - \frac{2\mathbb{E}[\hat{\Theta}_t \sin(2\hat{\Theta}_t)]}{\sinh(2(t+\delta))} \\ &= \frac{\cosh(2(t+\delta)) - 1}{a \sinh(2(t+\delta))} + \frac{\mathbb{E}[1 - \cos(2\hat{\Theta}_t) - 2a\hat{\Theta}_t \sin(2\hat{\Theta}_t)]}{a \sinh(2(t+\delta))} \\ &\leq \frac{1}{a}(t+\delta) + \frac{\mathbb{E}[1 - \cos(2\hat{\Theta}_t) - 2a\hat{\Theta}_t \sin(2\hat{\Theta}_t)]}{a \sinh(2(t+\delta))} \end{aligned}$$

where we have used the inequality

$$\frac{\cosh(t) - 1}{\sinh(t)} \leq t.$$

To proceed, we use a bound of the form

$$1 - \cos(2x) - 2ax \sin(2x) \leq \beta x^2$$

for some $\beta = \beta(a)$. Such bounds may be readily found, however, since we use different values of β for different ranges of κ , we proceed in the proof with a general value for β , and then specialize at the end.

Assuming such a bound, we get

$$\mathbb{E}[1 - \cos(2\hat{\Theta}_t) - 2a\hat{\Theta}_t \sin(2\hat{\Theta}_t)] \leq \beta \mathbb{E}[\hat{\Theta}_t^2]$$

and thus the differential equation becomes

$$\frac{d}{dt} \mathbb{E}[\hat{\Theta}_t^2] \leq \frac{1}{a}(\delta + t) + \frac{\beta \mathbb{E}[\hat{\Theta}_t^2]}{2a(\delta + t)}.$$

The solution to this differential inequality with the initial condition $\mathbb{E}[\hat{\Theta}_t^2] = 0$ is

$$\mathbb{E}[\hat{\Theta}_t^2] \leq \frac{2}{4a - \beta} [(t + \delta)^2 - \delta^{2-\beta/2a}(t + \delta)^{\beta/2a}].$$

We use this to obtain a bound on $\hat{T}_t(i\delta)$. This satisfies

$$d\hat{T}_t = -\frac{\sin(2\hat{\Theta}_t)}{\sinh(2(t+\delta))} dt,$$

and thus

$$d\hat{T}_t^2 = -\frac{2\hat{T}_t \sin(2\hat{\Theta}_t)}{\sinh(2(t+\delta))} dt.$$

By applying Cauchy-Schwartz, we have

$$\left| \frac{d}{dt} \mathbb{E}[\hat{T}_t^2] \right| \leq \mathbb{E} \left[\left| \frac{d}{dt} \hat{T}_t^2 \right| \right] = \frac{2\mathbb{E}[|\hat{T}_t \sin(2\hat{\Theta}_t)|]}{\sinh(2(t+\delta))} \leq \frac{2\|\hat{T}_t\|_2 \cdot \|\sin(2\hat{\Theta}_t)\|_2}{\sinh(2(t+\delta))}$$

which can be rearranged to

$$\left| \frac{d}{dt} \|\hat{T}_t\|_2 \right| = \left| \frac{\frac{d}{dt} \mathbb{E}[\hat{T}_t^2]}{2\|\hat{T}_t\|_2} \right| \leq \frac{\|\sin(2\hat{\Theta}_t)\|_2}{\sinh(2(t+\delta))}.$$

Before proceeding to integrate this to bound $\|\hat{T}_t\|_2$, we collect a few statements for later use.

Lemma 4.4. *Let*

$$U_t = \int_0^t \frac{\|\sin(2\hat{\Theta}_s)\|_2}{\sinh(2(s+\delta))} ds.$$

Then

$$\mathbb{E} \left[\left| \frac{d}{dt} \hat{T}_t^2 \right| \right] \leq 2U_t \frac{dU_t}{dt} = \frac{dU_t^2}{dt}.$$

Proof. Integrate the statement prior to the lemma to get $\|\hat{T}_t\|_2 \leq U_t$. Plugging this into the statement two above gives the result. \blacksquare

We now integrate to produce the bound on $\|\hat{T}_t\|_2$. Since $|\sin(x)| \leq 1 \wedge |x|$, we get

$$\begin{aligned} \left| \frac{d}{dt} \|\hat{T}_t\|_2 \right| &\leq \frac{1 \wedge 2\|\hat{\Theta}_t\|_2}{\sinh(2(t+\delta))} \\ &= \frac{1}{\sinh(2(t+\delta))} \wedge \frac{2^{3/2} [(t+\delta)^2 - \delta^{2-\beta/2a}(t+\delta)^{\beta/2a}]^{1/2}}{(4a-\beta)^{1/2} \sinh(2(t+\delta))}. \end{aligned}$$

The left term is small large values of t , whereas the right term is small for small values of t . We wish to swap the two bounds at t_0 chosen so that

$$\frac{4a-\beta}{8} = (t_0 + \delta)^2 - \delta^{2-\beta/2a}(t_0 + \delta)^{\beta/2a}.$$

We may bound this by

$$t_0 + \delta \leq \left(\frac{4a-\beta}{8} \right)^{1/2}.$$

By switching at this upper bound for t_0 , rather than at t_0 itself, we only hurt our estimate, and hence we get

$$\begin{aligned}
\|\hat{T}_t\|_2 &\leq \int_{\delta}^{(\frac{4a-\beta}{8})^{1/2}} \frac{2^{3/2} [s^2 - \delta^{2-\beta/2a} s^{\beta/2a}]^{1/2}}{(4a-\beta)^{1/2} \sinh(2s)} ds + \int_{(\frac{4a-\beta}{8})^{1/2}}^{\infty} \frac{1}{\sinh(2s)} ds \\
&\leq \int_{\delta}^{(\frac{4a-\beta}{8})^{1/2}} \frac{2^{3/2} s}{(4a-\beta)^{1/2} \sinh(2s)} ds + \int_{(\frac{4a-\beta}{8})^{1/2}}^{\infty} \frac{1}{\sinh(2s)} ds \\
&\leq \frac{1}{2} + \int_{(\frac{4a-\beta}{8})^{1/2}}^{\infty} \frac{1}{\sinh(2s)} ds \\
&= \frac{1}{2} + \frac{1}{2} \int_{(\frac{4a-\beta}{2})^{1/2}}^{\infty} \frac{1}{\sinh(s)} ds
\end{aligned}$$

where we used the bound $\sinh(s) \geq s$ for $s \geq 0$. This may be integrated exactly since

$$\int_{(\frac{4a-\beta}{2})^{1/2}}^{\infty} \frac{1}{\sinh(s)} ds = -\log \left(\tanh \sqrt{\frac{4a-\beta}{8}} \right),$$

yielding the following result.

Proposition 4.5. *Let \hat{T}_t be as above, then*

$$\|\hat{T}_t\|_2 \leq \frac{1}{2} - \frac{1}{2} \log \left(\tanh \sqrt{\frac{4a-\beta}{8}} \right),$$

or in the notation of Lemma 4.4,

$$U_{\infty} \leq \frac{1}{2} - \frac{1}{2} \log \left(\tanh \sqrt{\frac{4a-\beta}{8}} \right).$$

This produces the desired bound after reparametrization. We now show this bound also holds in the original parametrization.

Proposition 4.6.

$$\mathbb{E}[T_t^2] \leq \left[\frac{1}{2} - \frac{1}{2} \log \left(\tanh \sqrt{\frac{4a-\beta}{8}} \right) \right]^2.$$

Proof. First note

$$\begin{aligned}
\mathbb{E}[T_t^2] &= \mathbb{E}[\hat{T}_{\sigma^{-1}(t)}^2] \\
&= \mathbb{E} \left[\int_0^{\sigma^{-1}(t)} \frac{d\hat{T}_s^2}{ds} ds \right] \\
&= \mathbb{E} \left[\int_0^\infty \frac{d\hat{T}_s^2}{ds} \mathbf{1}_{\{s \leq \sigma^{-1}\}} ds \right] \\
&\leq \int_0^\infty \mathbb{E} \left[\left| \frac{d\hat{T}_s^2}{ds} \right| \right] ds.
\end{aligned}$$

Using Lemma 4.4 and Proposition 4.5, we obtain

$$\begin{aligned}
\mathbb{E}[T_t^2] &\leq \int_0^\infty \mathbb{E} \left[\left| \frac{d\hat{T}_s^2}{ds} \right| \right] ds \\
&\leq \int_0^\infty \frac{dU_s^2}{ds} ds \\
&= U_\infty^2 \leq \left[\frac{1}{2} - \frac{1}{2} \log \left(\tanh \sqrt{\frac{4a-\beta}{8}} \right) \right]^2.
\end{aligned}$$

■

4.3. Non-triviality results. Combining Lemma 4.1 with Proposition 4.6, we obtain the following result.

Theorem 4. *For any a and σ , if there exists $\beta > 0$ with*

$$1 - \cos(2x) - 2ax \sin(2x) \leq \beta x^2$$

and

$$2\sigma^2 \left[\frac{1}{2} - \frac{1}{2} \log \left(\tanh \sqrt{\frac{4a-\beta}{8}} \right) \right]^2 < 1,$$

then the associated parafermionic observable is not uniformly zero.

Proof.

$$\begin{aligned}
F(0) &= \lim_{t \rightarrow \infty} \lim_{\delta \downarrow 0} \mathbb{E}[e^{-2i\sigma T_t(i\delta)}] \\
&\geq 1 - 2\sigma^2 \limsup_{t \rightarrow \infty, \delta \downarrow 0} \mathbb{E}[T_t^2(i\delta)] \\
&\geq 1 - 2\sigma^2 \left[\frac{1}{2} - \frac{1}{2} \log \left(\tanh \sqrt{\frac{4a-\beta}{8}} \right) \right]^2.
\end{aligned}$$

■

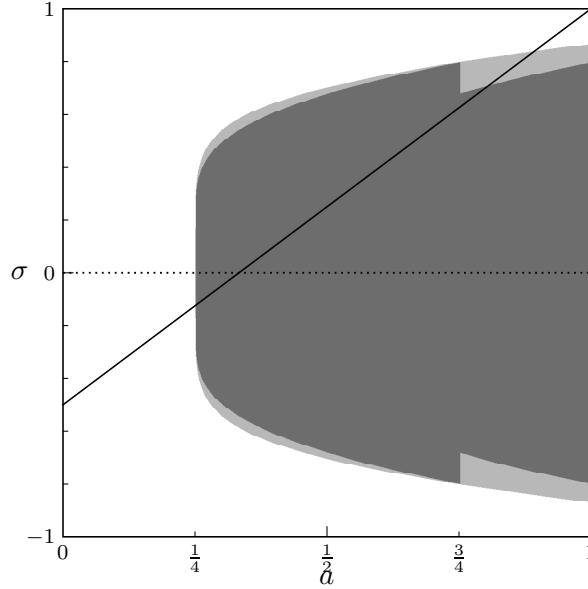


FIGURE 6. A numerical approximation of the set of a and σ (shown in gray) for which Theorem 4 can prove the parafermionic observable non-trivial. To produce this figure, the smallest value of β which satisfies the first inequality of Theorem 4 was numerically approximated for each choice of κ , and then the corresponding range of valid σ 's was computed for that combination of κ and β using the second inequality. The dark gray subset are those pairs for which non-triviality is established by the choices of β in Lemma 4.7. The solid line corresponding to the conformal pairing of a and σ .

Figure 6 illustrates a numerical approximation to the range for which this proof is valid.

We are most interested in the σ which yields the conformally invariant observable, $\sigma = (3a - 1)/2$, and hence we are most interested in finding β with

$$\frac{(3a - 1)^2}{8} \left[1 - \log \left(\tanh \sqrt{\frac{4a - \beta}{8}} \right) \right]^2 < 1.$$

We provide here explicit choices of β which allows us to show the associated parafermionic observables to be non-trivial over a wide range of κ . While it may be possible to expand the range slightly with better choices of β (particularly for large values of a) expanding it beyond $\kappa \in [0, 8]$, or indeed to any interval including either end point seems to require a different approach (mainly due to the coarseness of Lemma 4.1).

Lemma 4.7. *The follow hold:*

- for $a \in [1/4, 3/4]$,

$$1 - \cos(2x) - 2ax \sin(2x) \leq x^2,$$

- and for $a \in [1/2, 1]$.

$$1 - \cos(2x) - 2ax \sin(2x) \leq 2x^2.$$

Unfortunately, the inequality in Theorem 4 may not be solved in closed form forcing us to numerically approximate the range. Plots of the associated inequalities are given in Figure 7.

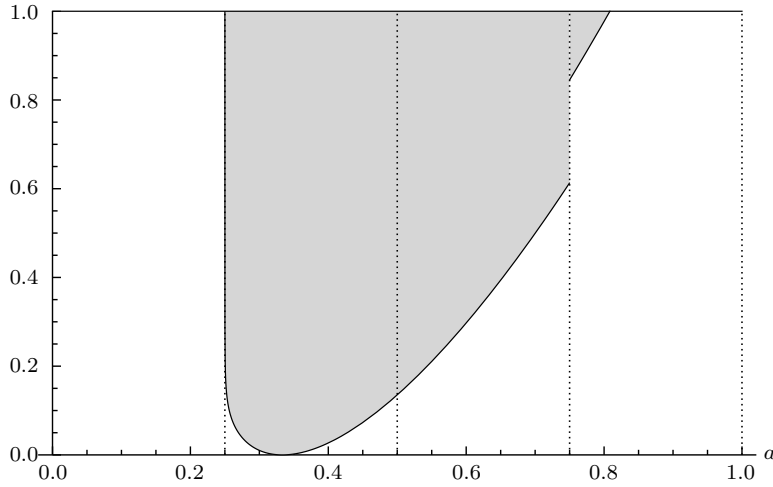


FIGURE 7. A graph showing the ranges over which the inequalities (illustrated as a shaded region) used in the proof of Theorem 5 holds. Vertical dashed lines indicate the ranges where the various choices of β hold.

Theorem 5. *Let a_0 be the solution to*

$$\frac{(3a-1)^2}{8} \left[1 - \log \left(\tanh \sqrt{\frac{4a-1}{8}} \right) \right]^2 = 1$$

nearest to $a = 1/4$ and a_1 be the solution to

$$\frac{(3a-1)^2}{8} \left[1 - \log \left(\tanh \sqrt{\frac{2a-1}{4}} \right) \right]^2 = 1$$

nearest to $a = 1$. Then, the conformal parafermionic observable is non-trivial for all $a \in (a_0, a_1)$ which is approximately $(0.2500000022 \dots, 0.8084748753 \dots)$, which corresponds to κ in $(2.4737936342 \dots, 7.9999999295 \dots)$ ¹.

¹These values were computed in Mathematica 7 with a working precision of 20 digits.

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